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THEORY OF AN ELASTIC LINEARLY REINFORCED COMPOSITE

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Elastic fibrous composites with an arbitrary cell microstructure are studied. A procedure is developed for determining the state of stress and the macroscopic properties of such materials. A rigorous foundation is given for the algorithms obtained. Results of computations are presented.

Composites with the simplest cell microstructure have been studied in [1], as well as by the method of [2] in [3]. General methods for investigating elastic inhomogeneous structures are contained in [4, 5].

1. Computational scheme for a composite. Formulation of the problem. Let us consider a three-dimensional isotropic medium reinforced by a doubly-periodic (in the sense of the geometry and elastic characteristics) system of groups of rectilinear fibers with cylindrical cavities (Fig. 1). The geometric and elastic properties of such a medium are described completely by the microstructure of the (fundamental) cell being duplicated periodically. Let us assume that the fibers are set in the medium with some transverse tension, identical at congruent points and constant along the fiber length. The connection between the medium and fiber is such that the force vector varies continuously during passage through the contact boundary, while the displacement vector undergoes a jump due to the transverse tension.

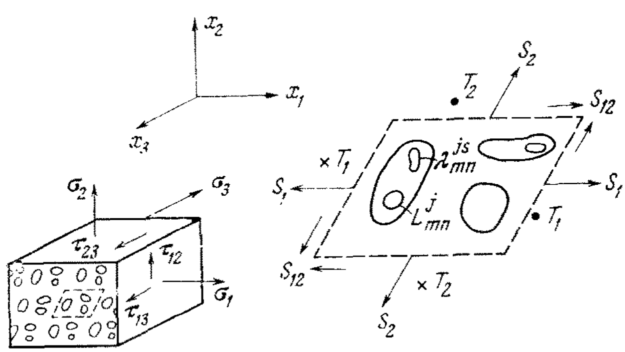

Fig. 1

Let us consider the following state of stress to be realized in the bulk

$$\begin{aligned}&\sigma_1 = \sigma_1(x_1, x_2), \quad \sigma_2 = \sigma_2(x_1, x_2), \quad e_3 = \text{const}\\ &\tau_{12} = \tau_{12}(x_1, x_2), \quad \tau_{13} = \tau_{13}(x_1, x_2), \quad \tau_{23} = \tau_{23}(x_1, x_2)\end{aligned} \tag{1.1}$$

Here and henceforth, $\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{13}, \tau_{23}$ are components of the stress tensor, $e_1, e_2, e_3, \gamma_{12}/2, \gamma_{13}/2, \gamma_{23}/2$ are components of the strain tensor.

The system (1.1) decomposes into two linearly independent states of stress "generalized plane strain" and "longitudinal shear"

$$\begin{aligned}&\sigma_1 = \sigma_1(x_1, x_2), \quad \sigma_2 = \sigma_2(x_1, x_2), \quad e_3 = \text{const}\\ &\tau_{12} = \tau_{12}(x_1, x_2), \quad \tau_{13} = \tau_{23} = 0\end{aligned} \tag{1.2}$$

$$\tau_{13} = \tau_{13}(x_1, x_2), \quad \tau_{23} = \tau_{23}(x_1, x_2), \quad \sigma_1 = \sigma_2 = e_3 = \tau_{12} = 0 \tag{1.3}$$

Let us use the regular Kolosov-Muskhelishvili functions $\varphi(z)$ and $\psi(z)$ in the complex variable $z = x_1 + ix_2$ which determine the stresses and displacements by means of the formulas [6]

$$\begin{aligned}&\sigma_1 + \sigma_2 = 2[\varphi'(z) + \overline{\varphi'(z)}]\\ &\sigma_2 - \sigma_1 + 2i\tau_{12} = 2[\bar{z}\varphi''(z) + \psi'(z)], \quad \varkappa = 3 - 4\nu\\ &2G(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - 2Ge_3\nu z\end{aligned} \tag{1.4}$$

to describe the state of stress (1.2).

Besides (1.4), we have from (1.2) and Hooke's law

$$\sigma_3 = 2G(1 + \nu)e_3 + \nu(\sigma_1 + \sigma_2), \quad w = e_3x_3 \tag{1.5}$$

Let us express the stresses and displacements correponding to (1.3) in terms of the regular function $F(z)$ [6]

$$\begin{aligned}&\tau_{13} - i\tau_{23} = G[2F'(\bar{z}) - ci\bar{z}]\\ &u = -cx_2x_3, \quad v = cx_1x_3, \quad w = F(z) + \overline{F(z)}\end{aligned} \tag{1.6}$$

In (1.4)–(1.6) we use the notation: u, v, w are the displacement vector componenets, G and ν are the shear modulus and Poisson's ratio of the material of the medium, and c is some real constant.

Therefore the problem of the theoretical description of a composite reduces to two-dimensional problems of elasticity theory. In conformity with this, let us consider a plane medium $x_3 = \text{const}$, reinforced by a doubly-periodic system of groups of foreign multiconnected inclusions.

Let ω_1 and ω_2 ($\operatorname{Im}\omega_1 = 0$, $\operatorname{Im}\omega_2/\omega_1 > 0$) be the fundamental periods of the structure. Each period parallelogram $\Pi_{m,n}$ ($m, n = 0, \pm 1, \ldots$) contains a group of inclusions with the elastic characteristics G_j and ν_j, filling the finite multiconnected domains $D^j_{m,n}$ ($j = 1, 2, \ldots, k$). Let $\lambda^{j,s}_{m,n}$ ($s = 1, 2, \ldots, r_j$) denote the hole outlines, $d^{j,s}_{m,n}$ the finite simply-connected continua bounded by the contours $\lambda^{j,s}_{m,n}$, $L^j_{m,n}$ the interface between the inclusion and the medium, and D the unbounded domain occupied by the medium (Fig. 1).

We use the following notation:

$$l_{m,n} = \bigcup_j L^j_{m,n}, \quad \Lambda^j_{m,n} = \bigcup_s \lambda^{j,s}_{m,n}, \quad \lambda_{m,n} = \bigcup_j \Lambda^j_{m,n}, \quad l^{*j}_{00} = l_{0,0} \setminus L^j_{0,0}$$

Let us assume that each of the contours $L_{m,n}^{j}$, $\lambda_{m,n}^{j,s}$ has a curvature satisfying the Hölder condition.

Let us assume that the average stresses over the fundamental periods S_1, S_2, S_{12}, T_1, T_2 exist in the medium (the forces S_1, S_2, S_{12} in Fig. 1 lie in the plane of the sketch, while the forces T_1, T_2 are perpendicular to it and the dot near the T_1, T_2 corresponds to the direction from the plane of the cross, and the cross along the direction to this plane) which assures double-periodicity of the stress tensor and quasiperiodicity of the function

$$g(z) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \quad \chi(z) = G\,[2\mathrm{Im}F(z) - c\,|z|^2/2] \qquad (1.7)$$

The identities

$$\begin{aligned} &g(z+\omega_1) - g(z) = -l_1\,[i\,(S_{12} + S_2\cos\theta) - S_2\sin\theta] \qquad (1.8)\\ &g(z+\omega_2) - g(z) = l_2\,[i(S_{12}\cos\theta + S_1) - S_{12}\sin\theta]\\ &\chi(z+\omega_1) - \chi(z) = l_1 T_2, \quad \chi(z+\omega_2) - \chi(z) = l_2 T_1\\ &l_1 = |\omega_1|, \quad l_2 = |\omega_2|, \quad \theta = \arg\omega_2 \end{aligned}$$

hence follow.

Integrating the first of the relationships (1.6) over the boundary of the fundamental period parallelogram, we obtain

$$c = 0 \qquad (1.9)$$

Quasiperiodicity of the functions $\varphi(z)$, $F(z)$, of the combination $\bar{z}\,\varphi'(z) + \psi(z)$ and of the displacements u, v, w follow from (1.4) – (1.6), (1.8) and (1.9) and the periodicity of the stress tensor.

Summarizing, we arrive at the problem of determining the functions $\varphi(z)$, $\psi(z)$, $F(z)$ and $\varphi_j(z)$, $\psi_j(z)$, $F_j(z)$ which are regular in the domains D and $D_{0,0}^{j}$ ($j = 1, 2, \ldots, k$), respectively, from the system of boundary conditions

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = \varphi_j(t) + t\overline{\varphi_j'(t)} + \overline{\psi_j(t)} \qquad (1.10)$$

$$\frac{1}{2G}\,[\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}] - \nu e_3 t =$$

$$\frac{1}{2G_j}\,[\varkappa_j\varphi_j(t) - t\overline{\varphi_j'(t)} - \overline{\psi_j(t)}] - \nu_j e_3 t + h_j(t), \quad t \in L_{0,0}^{j}$$

$$\varphi_j(t) + t\overline{\varphi_j'(t)} + \overline{\psi_j(t)} = C_{j,s} \qquad t \in \lambda_{0,0}^{j,s}$$

$$\begin{aligned} &F(t) + \overline{F(t)} = F_j(t) + \overline{F_j(t)} \qquad (1.11)\\ &G\,[F(t) - \overline{F(t)}] = G_j\,[F_j(t) - \overline{F_j(t)}] \qquad t \in L_{0,0}^{j}\\ &F_j(t) - \overline{F_j(t)} = iC_{j,s}^{*} \qquad t \in \lambda_{0,0}^{j,s}\\ &s = 1, 2, \ldots, r_j, \quad j = 1, 2, \ldots, k \end{aligned}$$

Here $h_j(t)$ are the known jumps in the displacement, and $C_{j,s}$ $C_{j,s}^{*}$ are constants to be determined.

Conditions for the existence of the given average stresses (1.8) in the medium must be appended to (1.10), (1.11).

2. Generalized plane strain of a composite. Let us use the results in [5, 7] to construct the general representations of the solution of boundary problem (1.10). Let us write

$$\varphi(z) = \frac{1}{2\pi i}\int_{l_{0,0}} p(t)\,\zeta(t-z)\,dt + Az \qquad (2.1)$$

$$\psi(z) = \frac{1}{2\pi i} \int_{l_{0,0}} [\varepsilon(t)\overline{p(t)} + \mu(t)\overline{q(t)} - \bar{t}p'(t)]\,\zeta(t-z)\,dt +$$

$$\frac{1}{2\pi i} \int_{l_{0,0}} p(t)\,\rho_1(t-z)\,dt + Bz, \quad z \in D$$

$$\varphi_j(z) = \frac{1}{2\pi i} \int_{L^j_{0,0}} \frac{q(t)}{t-z}\,dt + \frac{1}{2\pi i} \int_{\Lambda^j_{0,0}} \frac{\omega(t)}{t-z}\,dt$$

$$\psi_j(z) = \frac{1}{2\pi i} \int_{L^j_{0,0}} \frac{\alpha_j\overline{p(t)} + \beta_j\overline{q(t)} - \bar{t}q'(t)}{t-z}\,dt +$$

$$\frac{1}{2\pi i} \int_{\Lambda^j_{0,0}} \frac{\overline{\omega(t)} - \bar{t}\omega'(t)}{t-z}\,dt + \sum_{s=1}^{r_j} \frac{b_{j,s}}{z - z_{j,s}}, \quad z \in D^j_{0,0}$$

$$j = 1, 2, \ldots, k$$

$$\rho_1(z) = \sum_{m,n}{}' \left\{ \frac{\bar{P}}{(z-P)^2} - 2z\frac{\bar{P}}{P^3} - \frac{\bar{P}}{P^2} \right\}, \quad P = m\omega_1 + n\omega_2$$

$$p(t) = \{p_j(t) \quad t \in L^j_{0,0}\} \qquad q(t) = \{q_j(t), \ t \in L^j_{0,0}\}$$

$$\omega(t) = \{\omega_{j,s}(t), \ t \in \lambda^{j,s}_{0,0}\}$$

Here $\zeta(z)$ is the Weierstrass ζ-function [8], $\rho_1(z)$ is a special meromorphic function [9], $p(t)$, $q(t)$, $\omega(t)$ are functions to be determined, $z_{j,s} \in d^{j,s}_{0,0}$ and $b_{j,s}$ is a functional given by the relationships [7]

$$b_{j,s} = i \int_{\lambda^{j,s}_{0,0}} [\omega(t)\,\overline{dt} - \overline{\omega(t)}\,dt]$$

For piecewise-constant $\varepsilon(t) = \{\varepsilon_j, \ t \in L^j_{0,0}\}$, $\mu(t) = \{\mu_j, \ t \in L^j_{0,0}\}$ and the constants α_j and β_j ($j = 1, 2, \ldots, k$) we put [5]

$$\alpha_j = \frac{1+\varkappa}{\lambda_j - 1}, \quad \beta_j = \frac{1 + \varkappa_j\lambda_j}{1 - \lambda_j}$$

$$\varepsilon_j = \frac{\varkappa + \lambda_j}{\lambda_j - 1}, \quad \mu_j = \frac{(1+\varkappa_j)\lambda_j}{1-\lambda_j}, \quad \lambda_j = \frac{G}{G_j}$$

Using the properties of the function $\rho_1(z)$ [2]

$$\rho_1(z + \omega_\nu) - \rho_1(z) = \bar{\omega}_\nu\,\rho(z) + \gamma_\nu$$

$$\gamma_\nu = 2\rho_1\left(\frac{\omega_\nu}{2}\right) - \bar{\omega}_\nu\rho\left(\frac{\omega_\nu}{2}\right), \quad \nu = 1, 2$$

($\rho(z)$ is the Weierstrass ρ-function [8]), and the quasiperiodicity of the Weierstrass ζ-function, it is easy to show that the representation (2.1) assures a doubly-periodic distribution of the stresses in D.

The constants A and B in (2.1) are determined from the conditions (1.8). Evaluating the left sides in the first two identities in (1.8) by using the function $g(z)$ from (1.7) and (2.1), we obtain a system of two equations in $\operatorname{Re} A$ and B. Solving the system under the assumption that its compatibility condition

$$\mathrm{Im}\, a = 0 \tag{2.2}$$

is satisfied [5], we have

$$\begin{aligned}
&\mathrm{Re} A = \mathrm{Re} A_L + {}^1/_4\,(\langle\sigma_1\rangle + \langle\sigma_2\rangle) \\
&B = B_L + {}^1/_2\,(\langle\sigma_2\rangle - \langle\sigma_1\rangle + 2i\,\langle\tau_{12}\rangle) \\
&\mathrm{Re}\, A_L = \mathrm{Re}\left(\frac{\pi}{2S}\, a + \frac{\pi}{S}\, b - \frac{b\delta_1}{\omega_1}\right) \\
&B_L = \frac{\delta_1 - \gamma_1}{\omega_1} b - \frac{2\pi}{S}\,\mathrm{Re}\, b - \left(\frac{\pi}{S} - \frac{\delta_1}{\omega_1}\right)\mathrm{Re}\, a \\
&a = \frac{1}{2\pi i}\int\limits_{l_{0,0}} [\varepsilon(t)\overline{p(t)} + \mu(t)\overline{q(t)}]\,dt + \frac{1}{2\pi i}\int\limits_{l_{0,0}} p(t)\,\overline{dt} \\
&b = -\frac{1}{2\pi i}\int\limits_{l_{0,0}} p(t)\,dt, \quad \delta_1 = 2\zeta\left(\frac{\omega_1}{2}\right), \quad S = \omega_1\,\mathrm{Im}\,\omega_2
\end{aligned} \tag{2.3}$$

Here $\langle\sigma_1\rangle$, $\langle\sigma_2\rangle$, $\langle\tau_{12}\rangle$ are the average stresses on areas perpendicular to the coordinate axes. Formulas to transform the stress tensor components as the coordinate system rotates, are given in [6].

Since $\mathrm{Im} A$ does not influence the state of stress [6], let us assume $\mathrm{Im}\, A = 0$.

Now let us reduce the boundary value problem (1.10) to its equivalent system of Fredholm integral equations of the second kind in the functions $p\,(t)$, $q\,(t)$, $\omega\,(t)$. Using the Sokhotskii-Plemelj formulas [10] to pass to the limit values in (2.1), substituting them into (1.10) and assuming [7]

$$C_{j,s} = -\int\limits_{\lambda_{0,0}^{j,s}} \omega(t)\,ds \tag{2.4}$$

we obtain the desired system

$$\begin{aligned}
&p\,(t_0) - M_j\,\{p\,(t),\ q\,(t),\ \omega\,(t),\ t_0\} = P_j\,(t_0), \quad t_0 \in L_{0,0}^j \\
&q\,(t_0) - N_j\,\{p\,(t),\ q\,(t),\ \omega\,(t),\ t_0\} = Q_j\,(t_0), \quad t_0 \in L_{0,0}^j \\
&\omega\,(t_0) - R_{j,s}\,\{p\,(t),\ q\,(t),\ \omega\,(t),\ t_0\} = 0, \quad t_0 \in \lambda_{0,0}^{j,s} \\
&s = 1, 2, \ldots, r_j, \quad j = 1, 2, \ldots, k
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
M_j\{p(t), q(t)\,\omega(t), t_0\} = {} & \frac{1}{2\pi i}\int\limits_{L_{0,0}^j} \left\{p(t)\,d\ln\frac{\overline{\sigma(t-t_0)}}{\sigma(t-t_0)} + \right. \\
& \left.\frac{\mu_j}{\varepsilon_j}\, q(t)\,d\ln\frac{t-t_0}{\sigma(t-t_0)}\right\} - \frac{1}{2\pi i\varepsilon_j}\int\limits_{l_{0,0}} \overline{p(t)} \times \\
& d\,\{(t-t_0)\overline{\zeta(t-t_0)} - \overline{\zeta_1(t-t_0)}\} - \frac{1}{2\pi i}\int\limits_{l_{0,0}^{*j}} \left\{p(t) \times \right. \\
& \left.\left[\frac{\varepsilon(t)}{\varepsilon_j}\overline{\zeta(t-t_0)}\,\overline{dt} - \zeta(t-t_0)\,dt\right] + \frac{\mu(t)}{\mu_j}\, q(t)\,\overline{\zeta(t-t_0)}\,\overline{dt}\right\} + \\
& \frac{\mu_j}{\varepsilon_j}\frac{1}{2\pi i}\int\limits_{\Lambda_{0,0}^j} \frac{\omega(t)}{t-t_0}\,dt + t_0\left(1 + \frac{1}{\varepsilon_j}\right)\mathrm{Re}\, A_L + \frac{\bar{t}_0}{\varepsilon_j}\,\bar{B}_L
\end{aligned}$$

$$N_j\{p(t), q(t), \omega(t), t_0\} = -\frac{1}{2\pi i}\int\limits_{L_{0,0}^j} \left\{\frac{\alpha_j}{\beta_j}\, p(t)\,d\ln\frac{\sigma(t-t_0)}{t-t_0} + \right.$$

$$q(t)\,d\ln\frac{t-t_0}{\overline{t-t_0}} - \frac{\overline{q(t)}}{\beta_j}\,d\,\frac{t-t_0}{\overline{t-t_0}}\Big\} - \frac{1}{2\pi i}\times$$

$$\int\limits_{l^{*j}_{0,0}} \frac{\alpha_j}{\beta_j}\,p(t)\,\zeta(t-t_0)\,dt - \frac{1}{2\pi i}\int\limits_{\Lambda^j_{0,0}}\Big\{\omega(t)\Big[\frac{dt}{t-t_0} - \frac{\overline{dt}}{\beta_j\,\overline{(t-t_0)}}\Big] -$$

$$\frac{\overline{\omega(t)}}{\beta_j}\,d\,\frac{t-t_0}{\overline{t-t_0}}\Big\} + \frac{1}{\beta_j}\sum_{s=1}^{r_j}\frac{b_{j,s}}{\overline{t_0-z_{j,s}}} - \frac{\alpha_j}{\beta_j}\,t_0\,\mathrm{Re}\,A_L$$

$$R_{j,s}\{p(t), q(t), \omega(t), t_0\} = \frac{1}{2\pi i}\int\limits_{\Lambda^j_{0,0}}\Big\{\omega(t)\,d\ln\frac{t-t_0}{\overline{t-t_0}} -$$

$$\overline{\omega(t)}\,d\,\frac{t-t_0}{\overline{t-t_0}}\Big\} + \frac{1}{2\pi i}\int\limits_{L^j_{0,0}}\Big\{q(t)\Big(\frac{dt}{t-t_0} - \frac{\beta_j\,\overline{dt}}{\overline{t-t_0}}\Big) -$$

$$\alpha_j p(t)\,\frac{\overline{dt}}{\overline{t-t_0}} - \overline{q(t)}\,d\,\frac{t-t_0}{\overline{t-t_0}}\Big\} + \sum_{s=1}^{r_j}\frac{b_{j,s}}{\overline{t_0-z_{j,s}}} - C_{j,s}$$

$$\zeta_1(z) = -\int_0^z \rho_1(z)\,dz, \qquad \frac{d}{dz}\ln\sigma(z) = \zeta(z)$$

$$P_j(t_0) = t_0\Big(1+\frac{1}{\varepsilon_j}\Big)\frac{\langle\sigma_1\rangle + \langle\sigma_2\rangle}{4} + \frac{\bar t_0}{\varepsilon_j}\,\frac{\langle\sigma_2\rangle - \langle\sigma_1\rangle - 2i\langle\tau_{12}\rangle}{2} -$$

$$\frac{2G}{\varkappa+\lambda_j}\,[(\nu-\nu_j)\,e_3 t_0 + h_j(t_0)]$$

$$Q_j(t_0) = -\,t_0\,\frac{\alpha_j}{\beta_j}\,\frac{\langle\sigma_1\rangle + \langle\sigma_2\rangle}{4} - \frac{2G}{1+\varkappa_j\lambda_j}\,[(\nu-\nu_j)\,e_3 t_0 + h_j(t_0)]$$

Let us assume that the functions $h_j(t)$ are differentiable and their first derivatives satisfy the Hölder condition. For this it is sufficient that the solutions $p(t)$, $q(t)$, $\omega(t)$ possess the same property [10].

Let us note that every solution of the system (2.5) satisfies the compatibility condition (2.2) [5]. The operation of the given average stresses in the medium is thereby assured.

Using [7, 5], it can be proved that the system (2.5) is solvable for any right side, and always uniquely. Therefore, the system (2.5), in combination with the representation (2.1), governs the solution of the initial boundary value problem.

3. Longitudinal shear of a composite. Let us seek the solution of the boundary value problem (1.11) in the class of regular quasiperiodic functions. A doubly-periodic stress distribution will thereby be assured in D.

Let us set

$$F(z) = I(z),\quad z\in D;\quad F_j(z) = I(z),\quad z\in D^j_{0,0},\quad j=1,2,\ldots,k \qquad (3.1)$$

$$I(z) = \frac{1}{2\pi i}\int\limits_{l_{0,0}} im(t)\,\zeta(t-z)\,dt + \frac{1}{2\pi i}\int\limits_{\lambda_{0,0}} in(t)\,\zeta(t-z)\,dt + Ez$$

Here $m(t)$, $n(t)$ are real functions to be determined

$$m(t)=\{m_j(t),\ t\in L_{0,0}^{j}\},\quad n(t)=\{n_{j,s}(t),\ t\in\lambda_{0,0}^{j,s}\}$$

We find the constant E in (3.1) from conditions (1.8). Evaluating the left side of the last identity in (1.8) by using the function $\chi(z)$, we obtain from (1.7), (1.9) and (3.1)

$$\begin{aligned}
&\operatorname{Re}E=\frac{\langle\tau_{13}\rangle}{2G}+\frac{1}{2\pi\sin\theta}\operatorname{Im}\left[f\left(\frac{\delta_2}{l_2}-\frac{\delta_1}{l_1}\cos\theta\right)\right] \\
&\operatorname{Im}E=-\frac{\langle\tau_{23}\rangle}{2G}+\frac{\operatorname{Im}(f\delta_1)}{2\pi l_1} \\
&f=\int\limits_{l_{0,0}} m(t)\,dt+\int\limits_{\lambda_{0,0}} n(t)\,dt,\quad \delta_1=2\zeta\left(\frac{\omega_1}{2}\right),\quad \delta_2=2\zeta\left(\frac{\omega_2}{2}\right)
\end{aligned}\tag{3.2}$$

Here $\langle\tau_{13}\rangle$ and $\langle\tau_{23}\rangle$ are the average shear stresses on areas perpendicular to the coordinate axes x_1 and x_2. The connection between T_1, T_2 and $\langle\tau_{13}\rangle$, $\langle\tau_{23}\rangle$ is evident.

We satisfy the boundary conditions (1.11) because of the still unknown functions $m(t)$, $n(t)$.

Substituting the limit values of the function (3.1) evaluated by using the Sokhotskii-Plemelj formulas [10] into (1.11) and assuming as is done in [7] that

$$C_{j,s}^{*}=-\int\limits_{\lambda_{0,0}^{j,s}} n(t)\,ds \tag{3.3}$$

we arrive at a system of Fredholm integral equations of the second kind in $m(t)$ and $n(t)$

$$\begin{aligned}
&m(t_0)-G_*^{\,j}V\{m(t),n(t),t_0\}= \\
&\qquad\frac{G_*^{\,j}}{G}(\langle\tau_{13}\rangle\operatorname{Im}t_0-\langle\tau_{23}\rangle\operatorname{Re}t_0),\quad t_0\in L_{0,0}^{j} \\
&n(t_0)-V\{m(t),n(t),t_0\}+C_{j,s}^{*}= \\
&\qquad\frac{1}{G}(\langle\tau_{13}\rangle\operatorname{Im}t_0-\langle\tau_{23}\rangle\operatorname{Re}t_0),\quad t_0\in\lambda_{0,0}^{j,s} \\
&V\{m(t),n(t),t_0\}=\frac{1}{2\pi i}\int\limits_{l_{0,0}} m(t)\,d\ln\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}+\frac{1}{2\pi i}\int\limits_{\lambda_{0,0}} n(t)\times \\
&\qquad d\ln\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}+\frac{\operatorname{Im}(f\delta_1)\operatorname{Re}t_0}{\pi l_1}+\frac{\operatorname{Im}t_0}{\pi\sin\theta}\operatorname{Im}\left[f\left(\frac{\delta_2}{l_2}-\frac{\delta_1}{l_1}\cos\theta\right)\right] \\
&G_*^{\,j}=(G-G_j)/(G+G_j),\quad s=1,2,\ldots,r_j,\quad j=1,2,\ldots,k
\end{aligned}\tag{3.4}$$

The solutions $m(t)$ and $n(t)$ together with their first derivatives of the system (3.4) satisfy the Hölder condition [10].

Let us prove the solvability of the system obtained. To do this, let us consider the homogeneous system corresponding to (3.4) whose solution will be denoted by $m_0(t)=\{m_j^{\circ}(t),\ t\in L_{0,0}^{j}\}$ and $n_0(t)=\{n_{j,s}^{\circ}(t),\ t\in\lambda_{0,0}^{j,s}\}$. Let us ascribe the superscript zero to all the functionals and functions corresponding to this solution.

Evidently the homogeneous system corresponds to the boundary value problem (1.11) for average stresses $\langle\tau_{13}\rangle$ and $\langle\tau_{23}\rangle$ equal to zero.

Using the Dirichlet integral formula [8], it can be shown that the solution of this homogeneous boundary value problem is

$$\begin{aligned}
&F^{\circ}(z)=\text{const},\quad F_j^{\circ}(z)=\text{const},\quad j=1,2,\ldots,k \\
&\operatorname{Re}F^{\circ}(z)=\operatorname{Re}F_j^{\circ}(z),\quad G\operatorname{Im}F^{\circ}(z)=G_j\operatorname{Im}F_j^{\circ}(z)
\end{aligned}\tag{3.5}$$

Passing to the limit values in (3.1) and taking (3.5) into account here, we deduce that $n_{j,s}^{\circ}(t)$ are the boundary values of some functions regular in $d_{0,0}^{j,s}$. Hence, by virtue of the uniqueness theorem for the solution of the Dirichlet problem [8], it follows:

$$\overset{\circ}{n}_{j,s}(t) = \text{const}, \quad j = 1, 2, \ldots, k, \quad s = 1, 2, \ldots, r_j \tag{3.6}$$

According to the Sokhotskii-Plemelj formula, we have for (3.1)

$$F_j(t) - F(t) = im_j(t), \qquad j = 1, 2, \ldots, k$$

Substituting the function (3.5) in this latter relationship, we obtain

$$m_j^{\circ}(t) = \text{const}, \qquad j = 1, 2, \ldots, k \tag{3.7}$$

Evaluating (3.1) by using (3.7), (3.6), and taking (3.5) into account, we find

$$F^{\circ}(z) = 0, \ F_j^{\circ}(z) = 0, \ C_{j,s}^{*\circ} = 0, \ m_0(t) = 0, \ t \in l_{0,0} \tag{3.8}$$

Returning to the function $n_0(t)$ and using (3.3), (3.6) and (3.8), we have

$$n_0(t) = 0, \qquad t \in \lambda_{0,0} \tag{3.9}$$

Therefore, the system (3.4) is solvable for any right side, and always uniquely. The correctness of the representation (3.1) is thereby proved.

The system (3.4) and the representation (3.1) determine the solution of the boundary value problem (1.11) completely.

4. Macroscopic model of a composite. Definition. A macroscopic model of a composite will be understood to be a homogeneous anisotropic medium possessing the property that the average strains will agree for identical average stresses acting in the material and in the medium. We hence assume $h_j(t) = 0$ $(j = 1, 2, \ldots, k)$.

The average strains are determined as follows:

$$\begin{aligned}
\langle e_1 \rangle &= \frac{u(z+\omega_1) - u(z)}{l_1}, \qquad \langle e_3 \rangle = e_3 \\
\langle e_2 \rangle &= \frac{v(z+w_2) - v(z)}{l_2 \sin\theta} - \frac{v(z+\omega_1) - v(z)}{l_1} \operatorname{ctg}\theta \\
\langle \gamma_{12} \rangle &= \frac{u(z+\omega_2) - u(z)}{l_2 \sin\theta} + \frac{v(z+\omega_1) - v(z)}{l_1} - \\
&\quad \frac{u(z+\omega_1) - u(z)}{l_1} \operatorname{ctg}\theta \\
\langle \gamma_{13} \rangle &= \frac{w(z+\omega_1) - w(z)}{l_1} \\
\langle \gamma_{23} \rangle &= \frac{w(z+\omega_2') - w(z)}{l_2 \sin\theta} - \frac{w(z+\omega_1) - w(z)}{l_1} \operatorname{ctg}\theta
\end{aligned} \tag{4.1}$$

Introducing the average stress

$$\langle \sigma_3 \rangle = \frac{1}{S} \iint\limits_{\Pi_{0,0}} \sigma_3 dx_1 dx_2, \qquad S = \omega_1 \operatorname{Im} \omega_2 \tag{4.2}$$

and evaluating the right sides of (4.1), (4.2) by using the formulas (1.4) – (1.6), (2.1), (2.3) – (2.5), (3.1) – (3.4), we write (4.1) as

$$\langle e_k\rangle = c_{k1}\langle\sigma_1\rangle + c_{k2}\langle\sigma_2\rangle + c_{k3}\langle\sigma_3\rangle + c_{k6}\langle\tau_{12}\rangle, \qquad k=1,2,3 \quad (4.3)$$
$$\langle\gamma_{23}\rangle = c_{44}\langle\tau_{23}\rangle + c_{45}\langle\tau_{13}\rangle$$
$$\langle\gamma_{13}\rangle = c_{54}\langle\tau_{23}\rangle + c_{55}\langle\tau_{13}\rangle$$
$$\langle\gamma_{12}\rangle = c_{61}\langle\sigma_1\rangle + c_{62}\langle\sigma_2\rangle + c_{63}\langle\sigma_3\rangle + c_{66}\langle\tau_{12}\rangle$$

where c_{ij} $(i,\ j = 1, \ldots, 6)$ are macroscopic elastic parameters of the composite which depend only on the elastic characteristics of the composite constituents and on the geometry.

For example, for $i,\ j = 4,\ 5$ the expressions c_{ij} are

$$c_{44} = \frac{1}{G} - \frac{2\,\mathrm{Im}\,f_2}{S}, \qquad c_{55} = \frac{1}{G} - \frac{2\,\mathrm{Re}\,f_1}{S}$$
$$c_{45} = -\frac{2\,\mathrm{Im}\,f_1}{S}, \qquad c_{54} = -\frac{2\,\mathrm{Re}\,f_2}{S}$$
$$f = f_1\langle\tau_{13}\rangle + f_2\langle\tau_{23}\rangle, \qquad S = \omega_1 \mathrm{Im}\,\omega_2$$

Here the functionals f_1, f_2 are determined by the relationship (3.2) for $\langle\tau_{13}\rangle = 1$, $\langle\tau_{23}\rangle = 0$ and $\langle\tau_{23}\rangle = 1$, $\langle\tau_{13}\rangle = 0$, respectively.

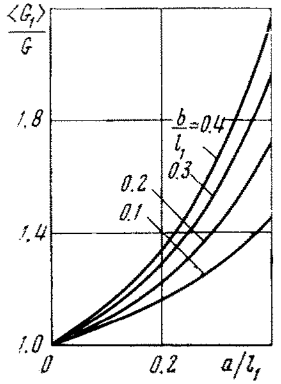

Fig. 2

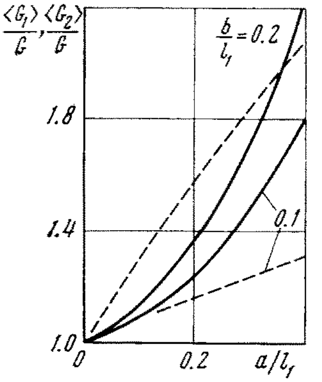

Fig. 3

It is important to emphasize that the c_{ij} depend on the solutions of the corresponding doubly-periodic problems in a functional manner, and it is sufficient to have just several functionals for the evaluation of c_{ij} This circumstance opens a path to diverse approximate approaches to the description of the macroproperties of composites.

The matrix of the coefficients c_{ij} $(i,\ j = 1,\ 2,\ 3,\ 6)$ is symmetric and energetically admissible [5]. It can be shown analogously to [5] that this last assertion extends to c_{ij} $(i,\ j = 4,\ 5)$. Therefore, it is admissible to treat (4.3) as the Hooke's law for the desired model medium.

As an illustration, let us consider the longitudinal shear of a composite of the boraluminum type (the ratio between the shear modulus of the fiber to the shear modulus of the medium equals 6.46) [11] with continuous fibers of elliptical cross section located at the vertices of a rectangular lattice $(\omega_1 = l_1,\ \omega_2 = il_2)$.

Presented in Figs. 2, 3 are curves of the change in the macroscopic parameters $(Gc_{55})^{-1} = \langle G_1\rangle / G$ and $(Gc_{44})^{-1} = \langle G_2\rangle / G$ (G is the shear modulus of the medium) as a function of the relative domain sizes a / l_1, b / l_1. In the case under consideration $c_{45} = c_{54} = 0$. Figure 2 corresponds to a square lattice $l_2 / l_1 = 1$, and Fig. 3 corresponds to a rectangular lattice $l_2 / l_1 = 0.5$ (solid lines correspond to $\langle G_1 \rangle$ and dashes to $\langle G_2 \rangle$). It is assumed that the semi-axes of the ellipse a and b are directed along the coordinate axes x_1 and x_2, respectively.

In the particular case $a = b, l_1 = l_2$ the results presented in Fig. 2 agree with the corresponding results in [3].

Let us note that in deriving (4.3) the stresses and displacements were averaged within the limits of a period parallelogram whose dimensions did not exceed 1 mm for the majority of composites. Hence, replacing a composite by a homogeneous anisotropic medium controlled by (4.3) in computational practice will apparently lead to satisfactory results.

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